

## BOREL OPEN COVERING OF HILBERT SCHEMES

MARGHERITA ROGGERO

ABSTRACT. Let  $p(t)$  be an admissible Hilbert polynomial in  $\mathbf{P}^n$  of degree  $d$  and Gotzmann number  $r$ . It is well known that  $\mathcal{H}\text{ilb}_{p(t)}^n$  can be seen as a closed subscheme of the Grassmannian  $\mathbb{G}(N, s)$ , where  $N = \binom{n+r}{n}$  and  $s = N - p(r)$ , hence, by Plücker embedding, it becomes a closed subset of a suitable projective space. Let us denote by  $\mathcal{B}$  the finite set of the Borel fixed ideals in  $k[X_0, \dots, X_n]$  generated by  $s$  monomials of degree  $r$ . We associate to every monomial ideal  $J \in \mathcal{B}$ , a Plücker coordinate  $p_J$ . If  $\mathcal{U}_J$  is the open subset of  $\mathbb{G}(N, s)$  given by  $p_J \neq 0$ , which is isomorphic to the affine space  $\mathbf{A}^{s(N-s)}$ , then  $\mathcal{H}_J = \mathcal{H}\text{ilb}_{p(t)}^n \cap \mathcal{U}_J$  is an open subset of  $\mathcal{H}\text{ilb}_{p(t)}^n$  and then can be seen as an affine subvariety of  $\mathbf{A}^{s(N-s)}$ . The main results obtained in this paper are the following:

- i)  $\mathcal{H}_J \neq \emptyset \Leftrightarrow \text{Proj}(k[X_0, \dots, X_n]/J) \in \mathcal{H}\text{ilb}_{p(t)}^n$ ;
- ii) if non-empty,  $\mathcal{H}_J$  parametrizes all the homogeneous ideals  $I$  such that the set  $\mathcal{N}J$  of the monomials not belonging to  $J$  is a basis of  $k[X_0, \dots, X_n]/I$  as a  $k$ -vector space;
- iii) the ideal defining  $\mathcal{H}_J$  as a subvariety of  $\mathbf{A}^{s(N-s)}$  (i.e. in “local” Plücker coordinates) is generated in degree  $\leq d + 2$ ;
- iv)  $\mathcal{H}_J$  can be isomorphically projected into a linear subspace of  $\mathbf{A}^{s(N-s)}$  of dimension  $\leq \sigma(N-s)$ , where  $\sigma$  is the number of minimal generators of the saturation  $J^{\text{sat}}$  of  $J$ ;
- v) up to changes of coordinates in  $\mathbf{P}^n$ , the open sets  $\mathcal{H}_J$  cover  $\mathcal{H}\text{ilb}_{p(t)}^n$ , namely:

$$\mathcal{H}\text{ilb}_{p(t)}^n = \bigsqcup_{\substack{g \in GL(n+1) \\ J \in \mathcal{B}}} g(\mathcal{H}_J).$$

## 1. INTRODUCTION

Let  $\mathcal{H}\text{ilb}_{p(t)}^n$  be the Hilbert scheme that parametrizes the subschemes in  $\mathbf{P}^n$  with Hilbert polynomial  $p(t)$ . We do not have to underline the importance of Borel ideals in most of the main general results obtained until now about Hilbert schemes, first and foremost the proof of the connectedness due to Robin Hartshorne ([6]). We also quote the paper [13] where Alyson Reeves using Borel ideals determines the “radius” of  $\mathcal{H}\text{ilb}_{p(t)}^n$ ; in that paper she posed the following as the first open question:

*Is the subset of Borel-fixed ideals on a component of  $\mathcal{H}\text{ilb}_{p(t)}^n$  enough to determine the component?*

We know that every irreducible component (and every intersection of irreducible components) of  $\mathcal{H}\text{ilb}_{p(t)}^n$  contains at least a Borel point (i.e. a point defined by a Borel ideal), because Galligo’s theorem states that every ideal can be deformed to a Borel ideal by a flat deformation. For this reason it is natural to take into consideration sets of points of  $\mathcal{H}\text{ilb}_{p(t)}^n$  that are in some sense “collected around” each Borel point.

To this purpose in this paper we give an explicit construction and investigate the properties of suitable open subsets  $\mathcal{H}_J$  of  $\mathcal{H}\text{ilb}_{p(t)}^n$  where  $J$  is Borel fixed ideal, based on the following algebraic condition on quotients:  $I$  defines a point in  $\mathcal{H}_J$  if the same set of monomials gives a  $k$ -vector basis for both the quotient modulo  $I$  and the quotient modulo  $J$ .

More generally, in §3 we consider for every monomial ideal  $J$  in  $\mathcal{P} := k[X_0, \dots, X_n]$  the family  $BSt(J)$  defined as follows:  $I \in BSt(J)$  if  $I$  is an homogeneous ideal in  $\mathcal{P}$  such that  $\mathcal{P} = I \oplus \langle \mathcal{N}J \rangle$ , as a  $k$ -vector space, where  $\mathcal{N}J$  is the sous-escalier of  $J$  that is the set of monomials that do not belong

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to  $J$ . If  $\mathcal{N}J$  is a finite set, this construction is clearly close to that of Border bases (see [7] and [14]). On the other hand, we can easily see, as a consequence of a well known property of initial ideals, that  $BSt(J)$  contains all the ideals  $I$  having  $J$  as its initial ideal with respect to some term order  $\preceq$  on the monomials. Can we find a characterization of the points in  $BSt(J)$  along the same lines as the theories of Gröbner and border bases?

In the last years several authors have been working on the family  $\mathcal{S}t_h(J, \preceq)$ , called *Gröbner stratum*, of all the ideals having the same initial ideal  $J$  with respect to a fixed term ordering  $\preceq$ , proving that  $\mathcal{S}t_h(J, \preceq)$  can be endowed of a structure of algebraic variety. We recall here [12], [15], [2] and mainly [1] by Giuseppa Carrà-Ferro, which is to our knowledge the first one. Usually the algebraic structure springs out of a procedure based on Buchberger's algorithm. However in [8] it is shown that most arguments involving the term ordering can be skipped out and replaced by some linear algebra tools.

Following the same idea that leads to the construction of  $\mathcal{S}t_h(J, \preceq)$ , in §3 we show that every ideal  $I$  in  $BSt(J)$  contains a unique set of homogeneous polynomials of the type  $G_I = \{f_\alpha = X^\alpha + \sum c_{\alpha\gamma} X^\gamma \mid X^\alpha \in G_J\}$  where  $G_J$  is the monomial basis of  $J$ ,  $X^\gamma \in \mathcal{N}J$  and  $c_{\alpha\gamma} \in k$  ( $c_{\alpha\gamma} \neq 0$  only if  $\deg(X^\gamma) = \deg(X^\alpha)$ ). In the case of a Gröbner stratum the set of polynomials  $G_I$  is the reduced Gröbner basis of  $I$ .

Even though in the present more general situation this idea does not function well, if we assume that  $J$  is a Borel ideal we are able to prove, only by using linear algebra, that the *Borel stratum*  $BSt(J)$  has a very natural and well defined structure of affine variety (Corollary 3.10).

In §4 the previously obtained results on  $BSt(J)$  are applied to the local study of the Hilbert scheme  $\text{Hilb}_{p(t)}^n$ . Firstly we choose for each point  $Z$  in  $\text{Hilb}_{p(t)}^n$  only one ideal defining it, which is not the most usual saturated ideal  $\mathcal{I}(Z)$ , but its truncation  $I := \mathcal{I}(Z)_{\geq r}$ , where  $r$  is the Gotzmann number of  $p(t)$ . Every ideal of this type is generated by  $s := N - p(r)$  forms of degree  $r$  (where  $N := \binom{n+r}{n}$ ). In this way we obtain the classical embeddings:  $\text{Hilb}_{p(t)}^n \hookrightarrow \mathbb{G}(s, N) \hookrightarrow \mathbf{P}^M$ , where the second one is the Plücker embedding.

We observe that each Plücker coordinate corresponds in a natural way to a monomial ideal  $J$ , generated by  $s$  monomials of degree  $r$ , namely generated by the linear space  $J_r \in \mathbb{G}(s, N)$ . Then we denote  $p_J$  a Plücker coordinate and by  $\mathcal{U}_J$  and  $\mathcal{H}_J$  respectively the open subsets of  $\mathbb{G}(s, V)$  and  $\text{Hilb}_{p(t)}^n$  given by  $p_J \neq 0$ : it is well known that  $\mathcal{U}_J \cong \mathbf{A}^{s(N-s)}$  and then  $\mathcal{H}_J \hookrightarrow \mathbf{A}^{s(N-s)}$  is an affine variety. The main aim of the present paper is to investigate general features of the affine varieties  $\mathcal{H}_J$  under the additional condition that  $J$  is a Borel ideal. This limiting hypothesis on  $J$  is of the utmost importance for the results to be obtained and repeatedly appears in the proofs. However, thanks to Galligo theorem on generic initial ideals, these special open subsets  $\mathcal{H}_J$  are sufficient for covering  $\text{Hilb}_{p(t)}^n$ , up to the action of the linear group  $GL(n+1)$ , that is up to changes of coordinates in  $\mathbf{P}^n$ .

Assuming that  $J$  is a Borel ideal, we first prove that  $\mathcal{H}_J$  is empty if and only if  $J$  does not define a point of  $\text{Hilb}_{p(t)}^n$ , while on the contrary the open subset  $\mathcal{H}_J$ , as an affine variety, is naturally isomorphic to the Borel stratum  $BSt(J)$  (Corollary 4.3 and Theorem 5.1).

Moreover, we are able to prove that the ideal  $\mathfrak{I}_J$  defining  $\mathcal{H}_J$  as a subvariety in  $\mathbf{A}^{s(N-s)}$  (that is in “local Plücker coordinates”  $p_\alpha/p_J$ ), is generated in a rather low degree. More precisely,  $\mathfrak{I}_J$  is generated in degree  $\leq d+2$ ,  $d$  being the dimension of the subschemes parametrized by  $\text{Hilb}_{p(t)}^n$ , that is the degree of the Hilbert polynomial  $p(t)$ . For instance, in the case of points, that is if  $\text{Hilb}_{p(t)}^n$  parametrizes 0-dimensional schemes, then  $\mathfrak{I}_J$  is generated in degrees  $\leq 2$ . A similar result in the 0-dimensional case has been presented by M.E. Alonso, J. Brachet and B. Mourrain at MEGA 2009.

Though a set of generators for the ideal  $\mathfrak{I}_J \subset k[C]$  (where  $|C| = s(N-s)$ ) can be obtained in a very simple way just by using linear algebra (they are some of the minors of a suitable matrix: see (5)) and

their degrees are reasonably low, the number  $s(N-s)$  of the involved variables is in general high. In the paper [8] an analogous situation has been analyzed for the Gröbner stratum  $\mathcal{St}_h(J, \preceq)$ : if  $J$  is generated by the  $s$  highest degree  $r$  monomials with respect to  $\preceq$ , then  $\mathcal{St}_h(J, \preceq) \cong \mathcal{U}_J$ . Under that stronger hypothesis,  $\mathfrak{J}_J$  turns out to be homogeneous with respect to a non-standard positive graduation on  $k[C]$ . This allows an embedding of  $\mathcal{H}_J$  in its Zariski tangent space at the point corresponding to  $J$  itself and the projection can be simply obtained by eliminating some variables appearing in the linear part of polynomials in  $\mathfrak{J}_J$  (see [15] and [2]). Simply by assuming that  $J$  is Borel, we may define a graduation on  $k[C]$ , in the same way as in the previous special case, and, again,  $\mathfrak{J}_J$  turns out to be homogeneous, but the graduation is not always positive.

However we can define a partial ordering on a suitable subset  $C''$  of the variables  $C$  that “partially” makes up for the possible non-positivity of the graduation. In this way we can prove that the variables  $C''$  can be eliminated. Therefore  $\mathcal{H}_J$  becomes a subvariety in the affine space  $\mathbf{A}^{\sigma(N-s)}$ , where  $\sigma$  is the number of minimal generators for the saturated ideal  $J^{sat}$  and it is often far lower than the number  $s$  of generators of  $J$  (Corollary 5.5).

## 2. NOTATION

Throughout the paper, we will consider the following general notation.

We work on a algebraically closed ground field  $k$  of characteristic 0.  $\mathcal{P} = k[X_0, \dots, X_n]$  is the polynomial ring in the set of variables  $X_0, \dots, X_n$  that we will often denote by the compact notation  $X$ . We will denote by  $X^\alpha$  the generic monomial in  $\mathcal{P}$ , where  $\alpha$  represents a multi-index  $(\alpha_0, \dots, \alpha_n)$ , that is  $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ .  $X^\alpha \mid X^\gamma$  means that  $X^\alpha$  divides  $X^\gamma$ , that is there exists a monomial  $X^\beta$  such that  $X^\alpha \cdot X^\beta = X^\gamma$ . If such monomial does not exist, we will write  $X^\alpha \nmid X^\gamma$ .

For every monomial  $X^\alpha \neq 1$  we set  $\min(X^\alpha) := \min\{i : X_i \mid X^\alpha\}$ . If  $\mathfrak{a}$  is a monomial ideal in  $\mathcal{P}$ ,  $G_{\mathfrak{a}}$  will denote its monomials basis and  $\mathcal{N}\mathfrak{a}$  its *sous-escalier* that is the set of monomials that do not belong to  $\mathfrak{a}$ . We will denote by  $\mathfrak{a}_m$  and  $\mathcal{N}\mathfrak{a}_m$  respectively the elements of degree  $m$  in either set.

We fix the following order on the set of variables  $X_0 < X_1 < \cdots < X_n$ . A monomial  $X^\beta$  can be obtained by a monomial  $X^\alpha$  through an *elementary move* if  $X^\beta X_i = X^\alpha X_j$  for some variables  $X_i \neq X_j$  or equivalently if there is a monomial  $X^\delta$  such that  $X^\alpha = X^\delta X_i$  and  $X^\beta = X^\delta X_j$ . We will denote an elementary move by  $X^\alpha \rightarrow X^\beta$  if  $X_i > X_j$  and by  $X^\alpha \leftarrow X^\beta$  if  $X_i < X_j$ . The transitive closure of the elementary moves gives a quasi-order on the set of monomials of any fixed degree that we will denote by  $\succ_B$ :

$$X^\alpha \succ_B X^\beta \iff X^\alpha \rightarrow X^{\gamma_1} \rightarrow \cdots \rightarrow X^{\gamma_t} \rightarrow X^\beta \text{ for some monomials } X^{\gamma_1}, \dots, X^{\gamma_t}.$$

An ideal  $\mathfrak{b} \subset k[X_0, \dots, X_n]$  is Borel-fixed if it is fixed under the action of the Borel subgroup of lower-triangular invertible matrices. We will use the following explicit characterization: a monomial ideal  $\mathfrak{b}$  is Borel fixed if and only if:  $X^\alpha \in \mathfrak{b}$  and  $X^\beta \rightarrow X^\alpha \implies X^\beta \in \mathfrak{b}$ .

Note that  $\prec_B$  agree with every term ordering  $\prec$  on  $\mathcal{P}$  such that  $X_0 \prec \cdots \prec X_n$ , that is:  $X^\alpha \prec_B X^\beta \implies X^\alpha \prec X^\beta$ .

## 3. THE FAMILY OF A BOREL IDEAL

In this initial section we want to generalize the construction of the homogeneous Gröbner stratum  $\mathcal{St}_h(J, \preceq)$  of a monomial ideal  $J$  in  $\mathcal{P}$  (see [1], [12], [15], [2], [8]). Roughly speaking,  $\mathcal{St}_h(J, \preceq)$  is an affine variety that parametrizes all the homogeneous ideals  $I \subset \mathcal{P}$  whose initial ideal with respect to a fixed term ordering  $\preceq$  is  $J$ . The main tools used in the construction of this family are reduced Gröbner bases and Buchberger algorithm. If  $G_J$  is the monomial basis of  $J$ , an ideal  $I$  belongs to  $\mathcal{St}_h(J, \preceq)$  if its reduced Gröbner basis is of the type  $\{f_\alpha = X^\alpha - \sum c_{\alpha\gamma} X^\gamma \mid X^\alpha \in G_J\}$  where  $c_{\alpha\gamma} \in k$ ,

$X^\gamma \in \mathcal{N}J$  and  $X^\gamma \prec X^\alpha$  (of the same degree). Due to the good properties of Gröbner bases, we can also observe that  $\mathcal{N}J$  is a basis of  $\mathcal{P}/I$  as a  $k$ -vector space.

We can now start from this last property and try to generalize the above construction, not considering any term ordering in  $\mathcal{P}$ . More explicitly, we want to study the family of all the homogeneous ideals  $I \subset \mathcal{P}$  such that  $\mathcal{N}J$  is a basis for the  $k$ -vector space  $\mathcal{P}/I$ . Every such ideal  $I$  has a special set of generators that looks like a reduced Gröbner basis. This remark motivates the following:

**Definition 3.1.** *Let  $J$  be a monomial ideal in  $\mathcal{P}$  with monomial basis  $G_J$ . We will call  $J$ -local set any set of homogeneous polynomials of the type:*

$$(1) \quad G = \left\{ f_\alpha = X^\alpha + \sum c_{\alpha\gamma} X^\gamma \mid X^\alpha \in G_J \right\} \text{ with } X^\gamma \in \mathcal{N}J \text{ and } c_{\alpha\gamma} \in k.$$

Moreover we will say that  $G$  is a  $J$ -local basis if  $\mathcal{N}J$  is a basis of  $\mathcal{P}/(G)$  as a  $k$ -vector space.

It is quite obvious from the definition, that the ideal generated by a  $J$ -local basis has the same Hilbert function than  $J$ .

The following examples shows that not every  $J$ -local set is also a  $J$ -local basis and that, more generally,  $J$ -local sets or even  $J$ -local bases do not have the good properties of Gröbner bases.

**Example 3.2.** *In  $k[x, y, z]$  let  $J = (xy, z^2)$  and  $I = (f_1 = xy + yz, f_2 = z^2 + xz)$ . The ideal  $I$  is generated by a  $J$ -local set. However  $J$  defines a 0-dimensional subscheme in  $\mathbf{P}^2$ , while  $I$  defines a 1-dimensional subscheme (because it contains the line  $x + z = 0$ ). Therefore,  $\{f_1, f_2\}$  is not a  $J$ -local basis because  $I$  and  $J$  do not have the same Hilbert function.*

Even when the ideal  $I$  is generated by a  $J$ -local set and they share the same Hilbert function, the  $J$ -local set is not necessarily a  $J$ -local basis, that is  $\mathcal{N}J$  is not necessarily a basis for  $\mathcal{P}/I$  as a  $k$ -vector space.

**Example 3.3.** *In  $k[x, y, z]$ , let  $J = (xy, z^2)$  and let  $I$  be the ideal generated by the  $J$ -local set  $\{g_1 = xy + x^2 - yz, g_2 = z^2 + y^2 - xz\}$ . It is easy to verify that both  $J$  and  $I$  are complete intersections of two quadrics and then they have the same Hilbert function. However,  $\mathcal{N}J$  is not a basis for  $\mathcal{P}/I$ : in fact  $zg_1 + yg_2 = x^2z + y^3 \in I$  is a sum of monomials in  $\mathcal{N}J$ , hence  $\{g_1, g_2\}$  is not a  $J$ -local basis.*

**Lemma 3.4.** *Let  $I$  and  $J$  be ideals in  $\mathcal{P}$ , with  $I$  homogeneous and  $J$  monomial.*

*If  $\mathcal{N}J$  is a basis for  $\mathcal{P}/I$  as a  $k$ -vector space, then  $I$  contains a unique  $J$ -local set  $G_I$ .*

*Proof.* Let  $X^\alpha$  be any monomial in the monomial basis  $G_J$  of  $J$ . By hypothesis, the class modulo  $I$  of  $X^\alpha$  can be written, in a unique way, as a  $k$ -linear combination  $\sum c_{\alpha\gamma} X^\gamma$  of monomials  $X^\gamma \in \mathcal{N}J$ . The difference of the two is then a polynomial  $f_\alpha \in I$  of the wanted type: note that  $f_\alpha$  must be homogeneous because  $I$  is homogeneous and  $\mathcal{N}J$  is a basis of the quotient.

The polynomials  $f_\alpha$  form the only  $J$ -local set  $G_I$  contained in  $I$  ■

When the monomial ideal  $J$  is the initial ideal of  $I$  with respect to a term ordering, then the reduced Gröbner basis of  $I$  is exactly the  $J$ -local set contained in  $I$  and it is also  $J$ -local basis for  $I$ . Hence  $I$  has a  $J$ -local basis which acts also as a set of generators. However in general the hypotheses of the previous lemma do not imply that the  $J$ -local set  $G_I$  contained in  $I$  is also a  $J$ -local basis; in fact the ideal generated by  $G_I$  can be strictly smaller than  $I$ .

**Example 3.5.** *In  $k[x, y, z]$  let  $J = (xy, z^2)$  and  $I = (f_1 = xy + yz, f_2 = z^2 + xz, f_3 = xyz)$ . Both  $I$  and  $J$  define 0 dimensional subschemes in  $\mathbf{P}^2$  of degree 4. In order to verify that  $\mathcal{N}J$  is a basis for  $k[x, y, z]/I$  it is sufficient to show that, for every  $m \geq 2$ , the  $k$ -vector space  $V_m = I_m + \mathcal{N}J_m = I_m + \langle x^m, y^m, x^{m-1}z, y^{m-1}z \rangle$  is equal to  $k[x, y, z]_m$ . For  $m = 2$ , this is obvious. Then, assume  $m \geq 3$ .*

First of all we observe that  $yz^2 = zf_1 - f_3 \in I$ : then  $V_m$  contains all the monomials  $y^{m-i}z^i$ . Moreover  $x^2y = xf_1 - f_3 \in I$  and  $xy^{m-1} = y^{m-2}f_1 - zy^{m-1} \in V_m$ : then  $V_m$  contains all the monomials  $x^{m-i}y^i$ . Finally, we can see by induction on  $i$  that all the monomials  $x^iz^{m-i}$  belongs to  $V_m$ . In fact as already proved,  $z^m \in V_m$ , hence  $x^{i-1}z^{m-i+1} \in V_m$  implies  $x^iz^{m-i} = x^{i-1}z^{m-i-1}f_2 - x^{i-1}z^{m-i+1} \in V_m$ .

However the  $J$ -local set  $G_I = \{f_1, f_2\}$  is not a  $J$ -local basis because it does not generate  $I$  (see Example 3.2).

We can recover most of the good properties of Gröbner bases in  $J$ -local bases, when  $J$  is a Borel fixed monomial ideal.

**Theorem 3.6.** *Let  $J$  be a Borel fixed monomial ideal in  $\mathcal{P}$  with monomial basis  $G_J$  and let  $I$  be a homogeneous ideal generated by a  $J$ -local set  $G_I = \{f_\alpha = X^\alpha + \sum c_{\alpha\gamma}X^\gamma \mid X^\alpha \in G_J\}$ .*

*Then  $\mathcal{N}J$  generate  $\mathcal{P}/I$  as a  $k$ -vector space, so that  $\dim_k I_m \geq \dim_k J_m$  for every  $m \geq 0$ .*

*Moreover:  $G_I$  is a  $J$ -local basis  $\iff I$  and  $J$  share the same Hilbert function.*

*Proof.* We will prove, by induction on  $m$ , that  $\mathcal{N}J_m$  generates  $(\mathcal{P}/I)_m$  namely that every monomial in  $J_m$  can be written modulo  $I$  as a sum of monomials in  $\mathcal{N}J_m$ . We will use every polynomial  $f_\alpha$  as a “rewriting law”: in fact, modulo  $I$ , every  $X^\alpha \in G_J$  can be replaced by  $-\sum c_{\alpha\gamma}X^\gamma$  with  $X^\gamma \in \mathcal{N}J$ .

If  $m = 0$  there is nothing to prove. Let us assume that  $m > 0$  and that the thesis holds in degree  $m - 1$ . Consider a monomial  $X^\beta \in J_m$ . If  $X^\beta \in G_J$ , we can immediately conclude using  $f_\beta \in G_I$ . On the contrary,  $X^\beta = X^\alpha X_i$  for some  $X^\alpha \in J_{m-1}$ : by the inductive hypothesis, we can rewrite  $X^\alpha$  modulo  $I$  by  $-\sum c_{\alpha\gamma}X^\gamma$  which is a sum of monomials of  $\mathcal{N}J_{m-1}$ . Then we can rewrite  $X^\beta$  by  $-\sum c_{\alpha\gamma}X^\gamma X_i$ . This is not always a sum of monomials in  $\mathcal{N}J_m$ , because some of the monomials  $X^\gamma X_i$  could belong to  $J_m$ . In case  $X^\gamma X_i \in G_J$ , we proceed as before rewriting it through the corresponding element in  $G_I$ . If, on the other hand,  $X^\gamma X_i = X^\eta X_j$  with  $X^\eta \in J_{m-1}$ , we can rewrite  $X^\eta$  using the inductive hypothesis. Again, some monomial of  $\mathcal{N}J_{m-1}$  multiplied by  $X_j$  could belong to  $J_m$  and so on.

This procedure will stop in a finite number of steps because  $J$  is Borel fixed. In fact, in this hypothesis,  $X^\gamma X_i = X^\eta X_j$ , with  $X^\gamma \in \mathcal{N}J_{m-1}$ ,  $X^\eta \in J_{m-1}$  implies  $X^\eta \rightarrow X^\gamma$ , hence  $X_i > X_j$ . ■

If the ideal  $J$  is not Borel fixed, the procedure of reduction described in the proof of the previous result could give rise to a “loop” without end.

**Example 3.7.** *Let us consider the ideals of Example 3.2. The monomial  $xyz$  can be reduced modulo  $I$  using  $f_1$  as  $xyz - zf_1 = -yz^2$ . This last monomial also belongs to  $J$  and can be reduced modulo  $I$  using  $f_2$  as  $-yz^2 + yf_2 = xyz$ , which is again the initial monomial.*

**Definition 3.8.** *For every Borel fixed ideal  $J$  in  $\mathcal{P}$ , the Borel stratum of  $J$  is the family, that we will denote by  $BSt(J)$ , of all homogeneous ideals  $I$  such that  $\mathcal{N}J$  is a basis of the quotient  $\mathcal{P}/I$  as a  $k$ -vector space.*

We can reformulate the previous results in the following way:

**Corollary 3.9.** *Let  $J$  be a Borel ideal. Then:*

$$I \in BSt(J) \iff I \text{ is generated by a } J\text{-local basis.}$$

Since a  $J$ -local basis for an ideal  $I$  is uniquely determined, we can give an explicit description of the family  $BSt(J)$  using this special set of generators.

Let us consider a new variable  $C_{\alpha\gamma}$  for every monomial  $X^\alpha$  in the monomial basis  $G_J$  of  $J$  and for every monomial  $X^\gamma \in \mathcal{N}J$  of the same degree as  $X^\alpha$ . We can denote by  $C$  the set of these new variables.

Moreover, let us consider the polynomials  $F_\alpha = X^\alpha + \sum C_{\alpha\gamma} X^\gamma \in k[X, C]$  and collect them in a set:

$$(2) \quad \mathcal{G} = \{F_\alpha = X^\alpha + \sum C_{\alpha\gamma} X^\gamma \mid X^\alpha \in G_J\}.$$

We can obtain every ideal  $I \in BSt(J)$  specializing (in a unique way) the variables  $C_{\alpha\gamma}$  in  $k$ , but not every specialization gives rise to an ideal in  $BSt(J)$ .

**Corollary 3.10.** *If  $J$  is a Borel fixed ideal in  $\mathcal{P}$ , the Borel stratum  $BSt(J)$  is an affine subvariety in  $\mathbf{A}^K$ , where  $K = \#(C)$ , that contains the origin, the point corresponding to  $J$  itself.*

*Proof.* Thanks to Theorem 3.6, a specialization  $C \mapsto c \in k$  transforms  $\mathcal{G}$  in a  $J$ -local basis  $G$  if and only if  $\dim_k((G)_m) \leq \dim_k(J_m)$  for every  $m \geq 0$ . We can then consider for each  $m$  the matrix  $A_m$  whose columns correspond to the degree  $m$  monomials in  $\mathcal{P}$  and whose rows contain the coefficients of those monomials in every polynomial of the type  $X^\gamma F_\alpha$  such that  $\deg(X^{\gamma+\alpha}) = m$ : every entry in  $A_m$  is either 0, 1 or one of the variables  $C$ . The ideal of  $BSt(J)$  in  $k[C]$  is then generated by all the minors in  $A_m$  of order  $\dim_k(J_m) + 1$  for every  $m \geq 0$ . ■

**Remark 3.11.** *Due to Gotzmann's persistence, in order to obtain a set of generators for the ideal of  $BSt(J)$  it will be sufficient to take into consideration the matrices  $A_m$  only for  $m \leq r + 1$ , where  $r$  is the Gotzmann number of the Hilbert polynomial of  $\mathcal{P}/J$ .*

#### 4. BOREL STRATA AND HILBERT SCHEMES

In this section we will apply the results obtained in the previous one and prove that Borel strata give in a very natural way an open covering for every Hilbert scheme of subschemes of  $\mathbf{P}^n$ .

First of all, let us recall how Hilbert schemes can be constructed and introduce some related notation. In the following,  $p(t)$  will be an admissible Hilbert polynomial in  $\mathbf{P}^n$  of degree  $d$  and  $\mathcal{Hilb}_{p(t)}^n$  will denote the Hilbert scheme parameterizing all the subschemes  $Z$  in  $\mathbf{P}^n$  with Hilbert polynomial  $p(t)$ . We will always denote by  $r$  the Gotzmann number of  $p(t)$ , that is the worst Castelnuovo-Mumford regularity for the subschemes parametrized by  $\mathcal{Hilb}_{p(t)}^n$ , by  $N$  the number  $\binom{n+r}{n}$ , and by  $s$  the number  $N - p(r)$ .

Every subscheme  $Z \in \mathcal{Hilb}_{p(t)}^n$  can be obtained by using many different ideals in  $\mathcal{P}$ , first of all the saturated ideal  $\mathcal{I}(Z) = \oplus_i H^0(\mathcal{I}_Z(i))$ , whose regularity is lower than or equal to  $r$ . Here we prefer to consider the “truncated” ideal  $I = \mathcal{I}(Z)_{\geq r}$ , which is generated by  $s$  linearly independent forms of degree  $r$  and so it is uniquely determined by a linear subspace of dimension  $s$  in the  $k$ -vector space  $V = \mathcal{P}_r$  of dimension  $N$ . Thus,  $\mathcal{Hilb}_{p(t)}^n$  can be embedded in the Grassmannian  $\mathbb{G}(s, V)$  of the  $s$ -dimensional subspaces of  $V$ . By abuse of notation, we will write  $I \in \mathbb{G}(s, V)$  if  $I$  is the ideal generated by  $I_r \in \mathbb{G}(s, V)$  and also  $I \in \mathcal{Hilb}_{p(t)}^n$  if moreover  $I$  defines a subscheme in  $\mathbf{P}^n$  with Hilbert polynomial  $p(t)$ .

Using Plücker coordinates,  $\mathbb{G}(s, V)$  becomes a closed subvariety of the projective space  $\mathbf{P}^M$ , where  $M + 1 = \binom{N}{s}$ : each Plücker coordinate corresponds to a set of  $s$  different elements in a given basis for  $V$ : the most natural basis to choose from is the one given by all the degree  $r$  monomials.

In the following, we will denote by  $\mathcal{Q}$  the set of the monomial ideals in  $\mathbb{G}(s, V)$  and by  $\mathcal{B}$  those that are Borel fixed.

There is an obvious 1 – 1 correspondence between Plücker coordinates and ideals in  $\mathcal{Q}$ : then we will denote by  $p_J$  the Plücker coordinate corresponding to  $J \in \mathcal{Q}$ .

If  $I \in \mathbb{G}(s, V)$ , due to Gotzmann's persistence we can say that  $I \in \mathcal{Hilb}_{p(t)}^n$  if and only if  $\dim_k(I_{r+1}) \leq s_1 := \binom{n+r+1}{n} - p(r + 1)$ . This condition directly gives equations defining  $\mathcal{Hilb}_{p(t)}^n$  as a subscheme of  $\mathbb{G}(s, V)$ .

**Remark 4.1.** A different choice of generators for  $V$  corresponds to a linear change of Plücker coordinates, but not to all of them. If for instance we consider a linear change of variables in  $\mathcal{P}$  and the basis of  $V$  given by the monomials in the new variables, the corresponding linear change of coordinates does not modify the equations of  $\mathbb{G}(s, V)$  in  $\mathbf{P}^M$ , while not every permutation of variables in  $\mathbf{P}^M$  does the same.

As well known,  $\mathbb{G}(s, V)$  can be covered by the open subsets  $\mathcal{U}_J$  given by  $p_J \neq 0$ ,  $J \in \mathcal{Q}$ . Each  $\mathcal{U}_J$  is isomorphic to the affine space  $\mathbf{A}^{s(N-s)}$ .

Since  $\text{Hilb}_{p(t)}^n \hookrightarrow \mathbb{G}(s, V)$  is in fact a closed embedding, the intersections  $\mathcal{H}_J = \mathcal{U}_J \cap \text{Hilb}_{p(t)}^n$  give an open covering of  $\text{Hilb}_{p(t)}^n$  by affine varieties. For the aim of the present paper we will consider a slightly different open covering of  $\text{Hilb}_{p(t)}^n$ : more precisely we will prove that  $\text{Hilb}_{p(t)}^n$  can be covered by the open subsets  $\mathcal{H}_J$ , where  $J$  varies among the Borel fixed ideals with Hilbert polynomial  $p(t)$ , and by the open subsets that we can obtain from these up homogeneous changes of coordinates in  $\mathcal{P}$ , that is under the action of the linear group  $GL(n+1)$ .

First of all we prove that if  $J \in \mathcal{B}$ , then  $\mathcal{H}_J$  is either empty or it is the Borel stratum  $BSt(J)$  defined in §3.

**Proposition 4.2.** Let  $J$  an ideal in  $\mathcal{B}$  and let  $I$  be a homogeneous ideal generated by a  $J$ -local set  $G_I$ .

If  $J \notin \text{Hilb}_{p(t)}^n$  then also  $I \notin \text{Hilb}_{p(t)}^n$ .

If, on the other hand,  $J \in \text{Hilb}_{p(t)}^n$ , then the following are equivalent:

- i)  $I \in \text{Hilb}_{p(t)}^n$
- ii)  $G_I$  is a  $J$ -local basis;
- iii)  $\mathcal{N}J_{r+1}$  is a basis for  $(\mathcal{P}/I)_{r+1}$  as a  $k$ -vector space;
- iv)  $\dim_k I_{r+1} = \dim_k J_{r+1}$ .

*Proof.* The first statement and “i)  $\Rightarrow$  ii)” are straightforward consequence of Theorem 3.6. The implications “ii)  $\Rightarrow$  iii)” and “iii)  $\Rightarrow$  iv)” are obvious. Finally, “iv)  $\Rightarrow$  i)” follows from Gotzmann persistence (see Remark 3.11). ■

We can summarize the previous proposition in terms of the open subsets  $\mathcal{H}_J$  as follows:

**Corollary 4.3.** If  $J \in \mathcal{B}$ , then:

- (1)  $\mathcal{H}_J = \emptyset \iff J \notin \text{Hilb}_{p(t)}^n$ .
- (2) If  $J \in \text{Hilb}_{p(t)}^n$ , then:  $I \in \mathcal{H}_J \iff I \in BSt(J)$ .

*Proof.* Both statements are straightforward consequence of Proposition 4.2 and of the definitions of  $\mathcal{H}_J$  and  $BSt(J)$ . ■

If  $J \in \mathcal{B}$ , the condition  $J \in \text{Hilb}_{p(t)}^n$  is sometimes expressed in literature saying that  $G_J$  is a Gotzmann set.

**Definition 4.4.** The Borel covering of  $\text{Hilb}_{p(t)}^n$  is the family containing the open subsets of the type  $\mathcal{H}_J$ , where  $J \in \mathcal{B} \cap \text{Hilb}_{p(t)}^n$ , and also containing the open subsets that can be obtained from these through a linear change of coordinates in  $\mathcal{P}$ , that is of the type  $g(\mathcal{H}_J)$ ,  $g \in GL(n+1)$ .

Now we prove that the expression “Borel covering” is correct.

**Proposition 4.5.** The “Borel covering” of  $\text{Hilb}_{p(t)}^n$  is in fact an open covering, that is:

$$\text{Hilb}_{p(t)}^n = \bigsqcup_{\substack{g \in GL(n+1) \\ J \in \mathcal{B}}} g(\mathcal{H}_J).$$

*Proof.* Let  $I \in \mathcal{Hilb}_{p(t)}^n$  be any ideal and  $\preceq$  any term ordering on monomials in  $\mathcal{P}$ . Due to Galligo Theorem ([3]), in general coordinates the initial ideal  $J$  of  $I$  is Borel fixed. Then, as well known,  $I$  and  $J$  have the same Hilbert polynomial. Since  $I$  is generated in degree  $r$ , which is its regularity, also  $J$  is generated in degree  $r$  and so  $J \in \mathcal{Hilb}_{p(t)}^n \cap \mathcal{B}$ . Then  $I$  belongs to the open subset  $\mathcal{H}_J$ .

Hence every ideal  $I \in \mathcal{Hilb}_{p(t)}^n$  belongs to some of the open subsets of the Borel covering.  $\blacksquare$

## 5. THE IDEAL DEFINING $\mathcal{H}_J$ IN $\mathbf{A}^{s(N-s)}$

In the present section we want to study the family that parametrizes all the ideals  $I \in \mathcal{Hilb}_{p(t)}^n$  with non-zero Plücker coordinate  $p_J$ , namely the open set  $\mathcal{H}_J$  of  $\mathcal{Hilb}_{p(t)}^n$ , and its properties as an affine variety. In the previous section it is proved that in the non-trivial case  $J \in \mathcal{Hilb}_{p(t)}^n$  and  $\mathcal{H}_J$  contains the same ideals than the Borel stratum of  $BSt(J)$ .

In §3 we proved that the Borel stratum  $BSt(J)$  can be seen as an affine subscheme in a suitable affine space  $\mathbf{A}^K$  (see Proposition 3.10) and in §4 that  $\mathcal{H}_J$  is a closed subscheme of the open subset  $\mathcal{U}_J \simeq \mathbf{A}^{s(N-s)}$  of the Grassmannian  $\mathbb{G}(s, V)$ , hence  $\mathcal{H}_J$  is an affine scheme. Now we will prove that in fact  $\mathcal{H}_J$  and  $BSt(J)$  are isomorphic.

**Theorem 5.1.** *If  $J \in \mathcal{Hilb}_{p(t)}^n \cap \mathcal{B}$ , then  $\mathcal{H}_J \simeq BSt(J)$  as affine varieties.*

*Moreover the ideal  $\mathfrak{I}_J$  defining  $\mathcal{H}_J$  as a subscheme of  $\mathbf{A}^{s(N-s)}$ , that is in “local” Plücker coordinates, is generated in degree  $\leq d + 2$ .*

*Proof.* First of all we verify that the number  $K$  of coordinates  $C$  is precisely  $s(N - s)$ . In fact if we write the set of homogeneous polynomials of (2) in the present hypothesis:

$$(3) \quad \mathcal{G} = \{F_\alpha = X^\alpha + \sum C_{\alpha\gamma} X^\gamma \mid X^\alpha \in G_J, X^\gamma \in \mathcal{N}J_r\}$$

we can see that  $|C| = |G_J| \cdot |\mathcal{N}J_r| = s(N - s)$ .

Moreover, every parameter  $C_{\alpha\gamma}$  is nothing else than the local Plücker coordinate  $p_\alpha/p_J$  where  $\alpha$  is the monomial ideal generated by  $G_\alpha = (G_J \cup \{X^\gamma\}) \setminus \{X^\alpha\}$ . Furthermore,  $\mathcal{H}_J$  as a subscheme of  $\mathbf{A}^{s(N-s)}$  and  $BSt(J)$  as a subscheme of  $\mathbf{A}^K$  are defined in the same way, that is imposing that the dimension in degree  $r + 1$  be  $s_1 := \dim_k(J_{r+1})$  (see Proposition 4.2).

For the second part of the thesis, we follow the proof of Proposition 3.10. Let us consider the matrix  $\mathcal{A} = A_{r+1}$ , whose columns correspond to the degree  $r + 1$  monomials in  $\mathcal{P}$  and whose rows contain the corresponding coefficients of the polynomials  $X_i F_\alpha$ . Every monomial in  $J_{r+1}$  can be written, not necessarily in a unique way, as a product of a monomial  $X^\alpha \in J_r$  and a variable  $X_i$ . Let us choose for each monomial in  $J_{r+1}$  one of this expressions  $X_i X^\alpha$  and order the columns of  $\mathcal{A}$  so that the first  $s_1$  columns correspond to the chosen forms  $X_i X^\alpha$  for the monomials of  $J_{r+1}$ , ordered in decreasing order with respect to the index  $i$ . Moreover let us order the rows of  $\mathcal{A}$  so that the first  $s_1$  of them contain the coefficient of  $X_i F_\alpha$  corresponding to and in the same order as the first  $s_1$  columns.

**Claim:** The square submatrix  $\mathcal{D}$  of  $\mathcal{A}$  given by the first  $s_1$  rows and columns has determinant equal to 1. More precisely, it is an upper triangular matrix of the following special form:

$$(4) \quad \mathcal{D} = \begin{pmatrix} I(n) & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & I(n-1) & \bullet & \bullet & \bullet & \bullet \\ \vdots & \vdots & \ddots & \bullet & \bullet & \bullet \\ 0 & \dots & 0 & I(k) & \bullet & \bullet \\ \vdots & \vdots & \vdots & 0 & \ddots & \bullet \\ 0 & \dots & \dots & \dots & 0 & I(0) \end{pmatrix}$$

where each  $I(k)$  stands for an identity matrix and corresponds to the group of rows and columns “ $X_k X^\alpha$ ”.



The coefficient of  $X_i X^\alpha$  in  $X_i F_\alpha$  is 1 and so the elements on the diagonal of  $\mathcal{D}$  are all 1. All the other non-zero entries in  $\mathcal{D}$  are coefficients of some monomial  $X^\gamma \in \mathcal{N}J_r$  that is variables  $C_{\alpha\gamma}$  such that  $X_i X^\gamma \in J$ , that is  $X_i X^\gamma = X_j X^\beta$  with  $X^\beta \in J_r$ . Then  $X^\gamma \leftarrow X^\beta$ , because  $J$  is Borel fixed. Hence  $X_i > X_j$ . This shows that we can find a non-constant entry in the row  $X_i F_\alpha$  only on columns corresponding to variables  $X_j$  strictly lower than  $X_i$  (i.e. where there is a “•” in the above picture). This completes the proof of the claim.

The ideal  $\mathfrak{J}_J$  of  $\mathcal{H}_J$  is then generated by the determinants of the submatrices of order  $s_1 + 1$  containing  $\mathcal{D}$ . Before computing those determinants we can row-reduce  $\mathcal{A}$  with respect to the first  $s_1$  rows. (Note that this is nothing else than the reduction procedure of Theorem 3.6 that uses the  $F_\alpha$  as rewriting lows).

In order to prove that  $\mathfrak{J}_J$  is generated in degree  $\leq d + 2$  we choose to order the rows and columns of  $\mathcal{A}$  in a special way: for every monomial in  $J_{r+1}$  we chose the form  $X_i X^\alpha$  with  $X_i = \min(X_i X^\alpha)$  and  $X^\alpha \in J_r$  (note that the hypothesis  $J$  Borel fixed allows this special choice). The key point in our argument is that for every  $X^\gamma \in \mathcal{N}J_r$ , the variable  $X_j = \min(X^\gamma)$  is lower than or equal to  $X_d$ , because a Borel ideal  $J$  can define a  $d$ -dimensional subscheme only if  $X_{d+1}^r \in J$ . This implies that we can find non-constant entries in  $\mathcal{A}$  only in columns correspondings to a  $j$ -block with  $j \leq d$ .

Thus  $\mathcal{A}$  assumes the following easier form:

$$(5) \quad \mathcal{A} = \left( \begin{array}{cccc|ccc} I(n, \dots, d+1) & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ 0 & I(d) & \bullet & \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & \ddots & \bullet & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & I(0) & \bullet & \dots & \bullet \\ \hline \star & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \star & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \end{array} \right)$$

where there is a big identity block  $I(n, \dots, d+1)$  corresponding to the variables  $X_n, \dots, X_{d+1}$  and “ $\star$ ” stands for a sequence of entries that are all 0 except at most a 1.

Then the ideal  $\mathfrak{J}_J$  is generated by polynomials of the type:

$$(6) \quad \det \left( \begin{array}{cccc|ccc} I(n, \dots, d+1) & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ 0 & I(d) & \bullet & \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & \ddots & \bullet & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & I(0) & \bullet & \dots & \bullet \\ \hline \star & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \end{array} \right).$$

After a partial row reduction, we obtain:

$$(7) \quad \det \left( \begin{array}{ccc|c} I(d) & \bullet & \bullet & \bullet \\ \vdots & \ddots & \bullet & \vdots \\ 0 & 0 & I(0) & \bullet \\ \hline \circ & \circ & \circ & \circ \end{array} \right)$$

where “ $\circ$ ” stands for a linear forms in  $k[C]$ . It is easy to controll (for instance by a rows reduction of the last row) that the determinant is a polynomial in  $k[C]$  of degree  $\leq d + 2$ .  $\blacksquare$

The ideal  $\mathfrak{J}_J$  in  $k[C]$  that defines  $\mathcal{H}_J$  is then generated in a quite small degree, but it lives in a polynomial ring with a very high number of variables. Now we will show that  $\mathcal{H}_J$  can be obtained by an ideal in a “smaller” polynomial ring. More precisely, we will prove that  $\mathcal{H}_J$  can be isomorphically projected in a linear space which is in general of far lower dimension than  $s(N - s)$ . This also will give an upper bound for the dimension of  $\mathcal{H}_J$  itself.

**Definition 5.2.** In the same hypothesis of Theorem 5.1, let  $E$  be the following set of monomials:

$$X^\beta \in E \iff X_0 X^\beta = X_i X^\alpha \text{ with } X_i \neq X_0 \text{ and } X^\alpha \in J_r.$$

Let us denote by  $C''$  the set of variables  $C_{\beta\gamma}$  with  $X^\beta \in E$  and by  $C'$  its complementary  $C \setminus C''$ .

In practice,  $E$  contains all the monomial of  $J_r$  except those that can be obtained multiplying by a suitable power of  $X_0$  the minimal generators of the ideal  $\mathfrak{b} = J^{\text{sat}}$ . Hence, every monomial in  $E$  is a product  $X_l X^\eta$  with  $l \neq 0$  and  $X^\eta \in \mathfrak{b}$ . Let us split the set of monomials  $E$  in the disjoint subsets  $E_l$  accordingly with the maximal  $X_l$  such that  $X^\alpha / X_l$  belongs to  $\mathfrak{b}$ .

It will be usefull to consider in  $C''$  the following quasi-order.

**Definition 5.3.** For every  $C_{\beta\gamma}, C_{\beta'\gamma'} \in C''$  we put  $C_{\beta\gamma} \succ C_{\beta'\gamma'}$  if  $X^\beta \in E_l, X^{\beta'} \in E_{l'}$  with  $l > l'$  or  $l = l'$  and  $X^\beta \succ_B X^{\beta'}$ .

**Theorem 5.4.** Let  $\mathfrak{b}$  be a saturated, Borel fixed ideal such that  $J = \mathfrak{b}_{\geq r} \in \text{Hilb}_{p(t)}^n$ . Then the ideal  $\mathfrak{I}_J \cap k[C']$  of  $k[C']$  defines an affine variety isomorphic to  $\mathcal{H}_J$ , that is  $\mathcal{H}_J$  can be isomorphically projected into the affine space  $\mathbf{A}^{|C'|}$  given by  $C'' = 0$ .

*Proof.* First of all, we prove that for every variable  $C_{\beta\gamma} \in C''$  we can find a polynomial  $F_{\beta\gamma} \in \mathfrak{I}_J$  in which  $C_{\beta\gamma}$  only appears as a degree 1 monomial and that the set of polynomials  $\{F_{\beta\gamma}\}$  allows the complete elimination of the variables  $C''$ .

Let us consider again the matrix  $\mathcal{A}$  defined in Theorem 5.1, but now we choose to write every monomial in  $J_{r+1}$  in the special form  $X_i X^\alpha$  where  $X_i$  is the maximal variable such that  $X^\alpha \in J_r$ .

By the definition, for every  $X^\beta \in E$  we have  $X_0 X^\beta = X_{i_0} X^{\alpha_0}$  where  $i_0 > 0$  and  $X_{i_0} F_{\alpha_0}$  corresponds to one of the first  $s_1$  rows in  $\mathcal{A}$ , that is to one of the rows of  $\mathcal{D}$ . Thus  $X_0 F_\beta$  corresponds to a row which is not in that first group of  $s_1$ . The polynomial  $F_{\beta\gamma}$  we are looking for is the determinant of the  $(s_1 + 1) \times (s_1 + 1)$  matrix  $\mathcal{D}'$  given by the first  $s_1$  rows and columns and by the row of  $X_0 F_\beta$  and the column of  $X_0 X^\gamma$ .

If  $X_n \neq X_{i_0}$ , every column in  $\mathcal{D}'$  that corresponds to a monomial of the type  $X_n X^\alpha$  contains only 0 entries, except 1 in the rows  $X_n F_\alpha$ : then we can exclude those columns and lines without modifying the determinant. Again, we can do the same for the group of columns  $X_i X^\alpha$  for every  $i = n - 1, \dots, i_0 + 1$  and also for the group of columns corresponding to  $X_i X^\alpha$  except  $X_{i_0} X^{\alpha_0}$ . Now  $F_{\beta\gamma}$  is the determinant of a matrix  $\mathcal{M}$  as in the following picture:

$$(8) \quad \begin{array}{c} X_{i_0} F^{\alpha_0} \\ X_{i_0+1} F^{\alpha'} \\ \dots \\ X_i F^\alpha \\ \dots \\ X_0 F^\beta \end{array} \left( \begin{array}{cccc|c} 1 & \bullet & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet & \bullet \\ \dots & & & & \\ \dots & 0 & 1 & \bullet & \bullet \\ \dots & \dots & & & \\ 0 & \dots & 0 & 1 & \bullet \\ \hline 1 & 0 & \dots & 0 & C_{\beta\gamma} \end{array} \right).$$

It remains to prove that the set of polynomials  $\{F_{\beta\gamma} / C_{\beta\gamma} \in C''\}$  allows the complete elimination of the set of variables  $C''$ . For this we observe that for every parameter  $C_{\alpha\delta} \in C''$  appearing as a  $\bullet$  in  $\mathcal{M}$  it holds  $C_{\beta\gamma} \succ C_{\alpha\delta}$  with respect to the quasi order of Definition 5.3.

In fact, as a consequence of the rule we adopted choosing the first rows,  $X^\beta \in E_{i_0}$ , while if  $X_i F_\alpha$  corresponds to one of the first  $s_1$  rows of  $\mathcal{M}$ , then  $X^\alpha \in E_l$  for some  $l \leq i \leq i_0$ . Then  $C_{\beta\gamma} \succ C_{\alpha\delta}$  if  $i > i_0$  and also if  $X^\alpha = X^{\alpha_0} \in E_{i_0}$  because  $X_0 X^\beta = X_{i_0} X^{\alpha_0}$  implies  $X^\beta \rightarrow X^{\alpha_0}$ . ■

**Corollary 5.5.** *Let  $J$  be a Borel ideal in  $\text{Hilb}_{p(t)}^n$ . If  $\sigma$  is the number of minimal generators for the saturated ideal  $J^{\text{sat}}$ , then:*

$$\mathcal{H}_J \hookrightarrow \mathbf{A}^{\sigma(N-s)}.$$

**Remark 5.6.** *If  $J$  is generated by an  $r$ -segment, that is by the  $s$  greatest monomials of degree  $r$  with respect to some term ordering  $\preceq$  on  $\mathcal{P}$ , then all the ideals in  $\mathcal{H}_J$  have initial ideal  $J$  with respect to  $\preceq$ , that is  $\mathcal{H}_J$  is isomorphic to the Gröbner stratum  $\text{St}_h(J, \preceq)$ . In this case we can apply a general result about Gröbner strata saying that the ideal  $\mathfrak{I}_J$  is homogeneous with respect to a positive graduation in  $k[C]$  and embed  $\mathcal{H}_J$  in its Zariski tangent space to the point  $J$  (see [15] and [2]). Hence, if  $J$  is a smooth point of  $\text{Hilb}_{p(t)}^n$ , then  $\mathcal{H}_J$  is isomorphic to an affine space, so that the component of  $\text{Hilb}_{p(t)}^n$  containing  $J$  is rational. (For properties of the points on  $\text{Hilb}_{p(t)}^n$  defined by an  $r$ -segment see also [16] and [17]).*

*Under the weaker hypothesis that  $J$  is a Borel ideal, anyway  $\mathfrak{I}_J$  turns out to be homogeneous. The graduation is given in the same way, that is  $k[C]$  is graduated over  $\mathbb{Z}^{n+1}$  and the degree of  $C_{\alpha\gamma}$  is  $\alpha - \gamma$ . However, the graduation is not always positive because it is possible to find non constant monomials with the same degree as the monomial 1. (In fact the graduation is positive if and only if there is a term ordering with respect to which  $J$  is of the above said type). The previous results allows an elimination of variables that partially substitutes the good properties of a positive graduation.*

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